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Asymptotic causality

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Abstract. For many physical systems it is clear that asymptotic causality must hold in some sense, without it being clear whether or not the system is strictly causal. In this paper the simple linear system is considered and it is shown that a quite general formulation of asymptotic causality (8) implies that the system is causal.

Let us consider the situation that arises frequently in physical problems, where an input $I(t)$ is converted linearly into an output $O(t)$, in a time-translational invariant way, so that

$$O(t) = \frac{1}{2\pi}(F*I)(t). \tag{1}$$

In this convolutional product it will be assumed that F is a tempered distribution and I an infinitely differentiable function of fast decrease ($F \in \mathscr{S}'$ and $I \in \mathscr{S}$). The output will then be an infinitely differentiable function bounded by a polynomial. A distribution is called causal (strictly causal) if it has support $[0, \infty)$, for then the output is related to the input in a causal way. For some physical systems it can be asserted that asymptotic or macro-causality must hold in some sense, without it being clear whether or not strict or micro-causality should hold. We have particularly in mind the related problem of asymptotic and local commutativity in quantum field theory, which it is hoped to discuss at a later date. In this paper an asymptotic causality condition is formulated and it is shown that, despite its apparent weakness, it in fact implies that the system is strictly causal. This problem has previously been discussed by Sucher (1959). Making much stronger assumptions than are made here, he has shown that a system that satisfies his asymptotic causality condition cannot be finitely acausal (F is finitely acausal if it has support $[a, \infty)$, $-\infty < a < 0$).

Before we define asymptotic causality we shall obtain a necessary and sufficient condition for F to be causal. If F is a tempered distribution with support $[0, \infty)$ ($F \in \mathscr{S}_+'$), then, as F is a tempered distribution, there exists a positive integer k such that

$$F(t) = t^k \chi(t) \tag{2}$$

with χ a distribution in $\mathscr{D}_{L_2}'$. From the assumed support property of F we see that the support of χ is $[0, \infty)$. It has been proved (Screaton 1969, appendix) that for such a distribution there exist constants C and L, such that

$$|\langle \chi, f \rangle| < C\left(\sum_{l=0}^{L} \int_0^\infty |D^l f|^2 \, dt \right)^{1/2} \tag{3}$$

for all $f \in \mathscr{D}_{L_2}$, where D denotes differentiation and $\langle \chi, f \rangle$ the action of the linear functional χ on the test function f. Thus

$$|\langle F, f \rangle| < C\left(\sum_{l=0}^{L} \int_0^\infty |D^l t^k f|^2 \, dt \right)^{1/2} \tag{4}$$

for all $f \in \mathscr{S}$. Using

$$\int_0^\infty |t^r D^s f|^2 \, dt = \int_0^\infty \frac{1}{1+t^2}(|t^r D^s f|^2 + |t^{r+1} D^s f|^2) \, dt$$

$$< \{(\sup_{0 \leqslant t < \infty} |t^r D^s f|)^2 + (\sup_{0 \leqslant t < \infty} |t^{r+1} D^s f|)^2\} \int_0^\infty \frac{dt}{1+t^2} \tag{5}$$

we see that for some C, N and L

$$|\langle F, f\rangle| < C \sum_{n=0}^{N} \sum_{l=0}^{L} \sup_{0 \leqslant t < \infty} |t^n D^l f|. \tag{6}$$

On the other hand, if F is a linear functional on $\mathscr{S}$ and (6) holds, then it is continuous and has support $[0, \infty)$, i.e. $F \in \mathscr{S}_+'$. Applying these considerations to the situation in which I is converted linearly into O in a translational invariant way, we see that a necessary and sufficient condition for I and O to be related as in (1), with F a causal tempered distribution, is

$$|O(t)| < C \sum_{n=0}^{N} \sum_{l=0}^{L} \sup_{-\infty < t' \leqslant t} |(t-t')^n D^l I(t')| \tag{7}$$

for some constants C, N and L, all $I \in \mathscr{S}$.

If a system is to be called asymptotically causal, then for each input it must tend to behave in a causal way as $t \to -\infty$. The relation (7) puts a bound on the behaviour of a causal output, in terms of the input. It would seem reasonable to take the requirement of asymptotic causality to be that, for each input I, there exist constants N_I, L_I such that the ratio of the output to

$$\sum_{n=0}^{N_I} \sum_{l=0}^{L_I} \sup_{-\infty < t' \leqslant t} |(t-t')^n D^l I(t')|$$

be well behaved as $t \to -\infty$ and that this behaviour be no worse than polynomial. We therefore say the following: *A system is asymptotically causal if, for each input $I(t)$, a polynomial $P_I(t)$ and constants N_I, L_I and T_I can be found such that the output $O(t)$ satisfies*

$$|O(t)| < P_I(t) \sum_{n=0}^{N_I} \sum_{l=0}^{L_I} \sup_{-\infty < t' \leqslant t} |(t-t')^n D^l I(t')|, \qquad t < T_I. \tag{8}$$

However,

$$|t^q (t-t')^r| \leqslant |t'^{q+r}|, \qquad 0 \geqslant t \geqslant t' > -\infty \tag{9}$$

and

$$|t'^r| = \sum_{s=0}^{r} {}^rC_s (t-t')^s t^{r-s};$$

thus (8) is equivalent to

$$|O(t)| < C_I \sum_{n=0}^{N_I} \sum_{l=0}^{L_I} \sup_{-\infty < t' \leqslant t} |t'^n D^l I(t')|, \qquad t < T_I \tag{8$'$}$$

where C_I, N_I, L_I and T_I are constants depending on I. It should be noted that there is no gain in generality obtained by replacing C_I by a polynomial in t.

To establish that F is causal when it is asymptotically causal, it will only be necessary to consider a sub-class of inputs

$$I(t) = \exp(-\delta t^2) \tag{10}$$

with δ real and positive. For this input

$$D^l I(t') = P_{l,\delta}(t') \exp(-\delta t'^2) \tag{11}$$

where $P_{l,\delta}$ is a polynomial of degree l. In the region $0 \geqslant t \geqslant t' > -\infty$

$$|t'^r \exp(-\delta t'^2)| < C_\delta(|t|^r + 1) \exp(-\delta t^2) \tag{12}$$

where here and subsequently C_δ denotes some constant whose value depends on δ. Using (11) and (12), we have

$$|t'^r D^l \exp(-\delta t'^2)| < C_\delta(|t|^{r+l} + 1) \exp(-\delta t^2), \qquad 0 \geqslant t \geqslant t' > -\infty. \tag{13}$$

From the asymptotic causality condition in the form (8$'$)

$$|O(t)| < C_\delta(|t|^{N_\delta} + 1) \exp(-\delta t^2); \qquad t < T_\delta, \qquad t \leqslant 0 \tag{14}$$

(N_δ is the sum of N_I and L_I of (8′)). As $O(t)$ is a continuous function and the right-hand side of (14) does not vanish, the bound can be extended to any finite region by suitably choosing C_δ. In particular, we have

$$|O(t)| < C_\delta(|t|^{N_\delta}+1)\exp(-\delta t^2), \qquad t \leqslant 0. \tag{15}$$

Defining $O_+(t)$ and $O_-(t)$ by

$$\begin{aligned} O_+(t) &= \theta(t)O(t) \\ O_-(t) &= \theta(-t)O(t) \end{aligned} \tag{16}$$

and regarding $O(t)$ as a distribution we have

$$O(t) = O_+(t)+O_-(t). \tag{17}$$

We denote the Fourier and Laplace transforms of a distribution $H(t)$ by $\tilde{H}(\xi)$ and $\tilde{H}(\xi+i\eta)$, respectively, where the Fourier transform $\tilde{f}(\xi)$ of a test function $f(t)$ is given by

$$\tilde{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\xi t)f(t)\,dt \tag{18}$$

and the Laplace transform $\tilde{H}(\xi+i\eta)$ is the Fourier transform of $e^{-\eta t}H(t)$. Then from (17) we see that

$$\tilde{O}(\xi) = \tilde{O}_+(\xi)+\tilde{O}_-(\xi). \tag{19}$$

As $O_\pm$ are tempered distributions with support $[0, \pm\infty)$, their Laplace transforms $\tilde{O}_\pm(\omega)$ are regular in the half-planes $\operatorname{Im}\omega \gtrless 0$ respectively, with the $\mathscr{S}'$ boundary values $\tilde{O}_\pm(\xi)$, and are bounded by

$$|\tilde{O}_\pm(\omega)| < C_\delta(1+|\operatorname{Im}\omega|^{-L_\delta})(1+|\omega|^{M_\delta}) \tag{20}$$

in their respective domains of regularity, for suitably chosen C_δ, L_δ and M_δ (see, for example, theorems 2–8, 2–9 and 2–10 of Streater and Wightman 1964). The bound (15) ensures that the Laplace transform of O_- exists for all ω and is given by

$$\tilde{O}_-(\omega) = \frac{1}{2\pi}\int_{-\infty}^{0} \exp(i\xi t)\exp(-\eta t)O(t)\,dt, \qquad \omega = \xi+i\eta. \tag{21}$$

But

$$|\tilde{O}_-(\omega)| \leqslant \frac{1}{2\pi}\int_{-\infty}^{0} \exp(-\eta t)|O(t)|dt < C_\delta\int_{-\infty}^{0} (|t|^{N_\delta}+1)\exp\left\{-\delta\left(t+\frac{\eta}{2\delta}\right)^2\right\}\exp\left(\frac{\eta^2}{4\delta}\right)dt. \tag{22}$$

If we write

$$|t|^{N_\delta} = (-)^{N_\delta}\sum_{n=0}^{N_\delta} {}^{N_\delta}C_n\left(t+\frac{\eta}{2\delta}\right)^n\left(-\frac{\eta}{2\delta}\right)^{N_\delta-n} \tag{23}$$

and observe that

$$\left|\int_{-\infty}^{0}\left(t+\frac{\eta}{2\delta}\right)^n\exp\left\{-\delta\left(t+\frac{\eta}{2\delta}\right)^2\right\}dt\right| < \int_{-\infty}^{\infty}|t|^n\exp(-\delta t^2) < C_\delta, \qquad n \leqslant N_\delta \tag{24}$$

it follows that the integral (21) is absolutely and uniformly convergent on any compact set of the η's, being bounded by $C_\delta(|\eta|^{N_\delta}+1)\exp(\eta^2/4\delta)$. Hence $\tilde{O}_-(\omega)$ is an entire function and satisfies

$$|\tilde{O}_-(\omega)| < C_\delta(|\eta|^{N_\delta}+1)\exp\left(\frac{\eta^2}{4\delta}\right). \tag{25}$$

The Fourier transforms of O, F and I are related by

$$\tilde{O}(\xi) = \tilde{F}(\xi)\tilde{I}(\xi) \tag{26}$$

on $\mathscr{S}$. With the choice (10) of I this becomes

$$\tilde{O}(\xi) = (4\pi\delta)^{-1/2}\exp\left(-\frac{\xi^2}{4\delta}\right)\tilde{F}(\xi). \tag{27}$$

The functions $\tilde{O}_{\pm}(\omega)$ are regular in the upper half-plane and $\tilde{I}(\xi)$ has an analytic continuation into this region, and so one expects to be able to continue $\tilde{F}(\xi)$. We show that

$$\tilde{G}(\omega) = (4\pi\delta)^{1/2}\exp\left(\frac{\omega^2}{4\delta}\right)\{\tilde{O}_{+}(\omega)+\tilde{O}_{-}(\omega)\}, \qquad \eta > 0 \tag{28}$$

has the boundary value $\tilde{F}(\xi)$:

$$\lim_{\epsilon \to 0_+} \tilde{G}(\xi+\mathrm{i}\epsilon) = \tilde{F}(\xi) \text{ on } \mathscr{D}. \tag{29}$$

As $\tilde{O}_{-}(\omega)$ has the $\mathscr{S}'$ boundary value $\tilde{O}_{-}(\xi)$ from the lower half-plane, we have

$$\langle \tilde{O}_{-}(\xi), h(\xi)\rangle = \lim_{\epsilon \to 0_+}\int_{-\infty}^{\infty} \tilde{O}_{-}(\xi-\mathrm{i}\epsilon)h(\xi)\,\mathrm{d}\xi \tag{30}$$

where $h \in \mathscr{S}$. Since $\tilde{O}_{-}(\omega)$ is regular on the real axis, we see that when h has compact support ($h \in \mathscr{D}$) the sign of ϵ in the integrand can be changed: thus

$$\tilde{O}_{-}(\xi) = \lim_{\epsilon \to 0_+} \tilde{O}_{-}(\xi+\mathrm{i}\epsilon) \text{ on } \mathscr{D}. \tag{31}$$

Now, for $h \in \mathscr{D}$,

$$\begin{aligned}\lim_{\epsilon \to 0_+}\int_{-\infty}^{+\infty}\exp\left\{\frac{(\xi+\mathrm{i}\epsilon)^2}{4\delta}\right\}\tilde{O}_{\pm}(\xi+\mathrm{i}\epsilon)h(\xi)\,\mathrm{d}\xi &= \lim_{\epsilon \to 0_+}\left\langle \tilde{O}_{\pm}(\xi+\mathrm{i}\epsilon)-\tilde{O}_{\pm}(\xi), \exp\left\{\frac{(\xi+\mathrm{i}\epsilon)^2}{4\delta}\right\}h(\xi)\right\rangle \\ &+ \lim_{\epsilon \to 0_+}\left\langle \tilde{O}_{\pm}(\xi), \left[\exp\left\{\frac{(\xi+\mathrm{i}\epsilon)^2}{4\delta}\right\}-\exp\left(\frac{\xi^2}{4\delta}\right)\right]h(\xi)\right\rangle \\ &+ \left\langle \exp\left(\frac{\xi^2}{4\delta}\right)\tilde{O}_{\pm}(\xi), h(\xi)\right\rangle.\end{aligned} \tag{32}$$

The two limits on the right-hand side are zero, the first because the convergence of $\tilde{O}_{\pm}(\xi+\mathrm{i}\epsilon)$ to $\tilde{O}_{\pm}(\xi)$ must be uniform on the bounded set of $\mathscr{D}$ (weak convergence in $\mathscr{D}'$ implies strong convergence) and the second because of the continuity of distributions. Hence

$$\lim_{\epsilon \to 0_+} \tilde{G}(\xi+\mathrm{i}\epsilon) = (4\pi\delta)^{1/2}\exp\left(\frac{\xi^2}{4\delta}\right)\tilde{O}(\xi) = \tilde{F}(\xi) \text{ on } \mathscr{D}. \tag{33}$$

The function $\tilde{G}(\omega)$ has a unique boundary value and is therefore independent of δ. From (20) and (25)

$$|\tilde{G}(\omega)| < C_\delta\left\{(1+\eta^{-L_\delta})(1+|\omega|^{M_\delta})\left|\exp\left(\frac{\omega^2}{4\delta}\right)\right| + (1+\eta^{N_\delta})\exp\left(\frac{\xi^2}{4\delta}\right)\right\}, \qquad \eta > 0. \tag{34}$$

The distribution F can be split into the pieces $F_{\pm}$ with support $[0, \pm\infty)$, so that

$$\begin{aligned} F(t) &= F_{+}(t)+F_{-}(t) \\ \tilde{F}(\xi) &= \tilde{F}_{+}(\xi)+\tilde{F}_{-}(\xi)\end{aligned} \tag{35}$$

on $\mathscr{S}$. The Laplace transforms $\tilde{F}_{\pm}(\omega)$ are regular in the half-planes $\eta \gtrless 0$, with $\mathscr{S}'$ boundary values $\tilde{F}_{\pm}(\xi)$, and satisfy the bounds

$$|\tilde{F}_{\pm}(\omega)| < C(1+|\eta|^{-L})(1+|\omega|^{M}), \qquad \eta \gtrless 0. \tag{36}$$

The function $\tilde{G}(\omega)-\tilde{F}_+(\omega)$ is regular in the upper half-plane and has $\tilde{F}_-(\xi)$ as its boundary values on $\mathscr{D}$. Thus $\tilde{F}_-(\omega)$ is an entire function (Streater and Wightman 1964, theorem 2–16), with $\tilde{G}(\omega)-\tilde{F}_+(\omega)$ providing the analytic continuation into the upper half-plane. Using (34) and (36), we see that $\tilde{F}_-(\omega)$ has the following properties:

(i) $\tilde{F}_-(\omega)$ is entire;

(ii) $|\tilde{F}_-(\omega)| < C(1+|\eta|^{-L})(1+|\omega|^M), \qquad \eta<0;$

(iii) $|\tilde{F}_-(\omega)| < C_\delta(1+\eta^{-L_\delta})(1+|\omega|^{M_\delta})\exp\left(\frac{|\omega|^2}{4\delta}\right), \qquad \eta>0$ all δ;

(iv) $|\tilde{F}_-(\mathrm{i}\eta)| < C(1+|\eta|^N)$.

(ii) is just (36). (iii) follows by choosing C_δ large enough, L_δ the maximum of the L_δ and L of (34) and (36), and M_δ as in (34). Using (34) with a fixed δ (δ' say) and noticing that $\tilde{F}_-(\omega)$ is regular at the origin, we have (iv) with C chosen large enough and N the maximum of $M-L$ and $N_{\delta'}$.

The constraints (i)–(iv) ensure that $\tilde{F}_-(\omega)$ is a polynomial. To prove this, it only has to be shown that $\tilde{F}_-(\omega)$ is polynomially bounded. In the region $\eta \leqslant -1$ condition (ii) gives the polynomial bound

$$|\tilde{F}_-(\omega)| < C(1+|\omega|^M), \qquad \eta \leqslant -1. \tag{37}$$

On the lines ($\eta = -1$), ($\xi = 0$, $\eta \geqslant -1$) (ii) and (iv) give polynomial bounds. To obtain the polynomial bound in the whole region $\eta \geqslant -1$, the Phragmen–Lindelof theorem will be used.

Above and on the line $\eta = 1$, condition (iii) gives

$$|\tilde{F}_-(\omega)| < C_\delta(1+|\omega|^{M_\delta})\exp\left(\frac{|\omega|^2}{4\delta}\right), \qquad \eta \geqslant 1. \tag{38}$$

By suitably choosing C_δ, we show that this bound can be extended to the region $\eta \geqslant -1$. A polynomial P_δ, of degree M_δ, with no zeros in $-1 \leqslant \eta \leqslant 1$, can be found, so that

$$\left|\frac{\exp(-\omega^2/4\delta)\tilde{F}_-(\omega)}{P_\delta(\omega)}\right| \leqslant 1 \tag{39}$$

on the lines ($\eta = \pm 1$), ($\xi = 0$, $-1 \leqslant \eta \leqslant 1$), and

$$\left|\frac{\exp(-\omega^2/4\delta)\tilde{F}_-(\omega)}{P_\delta(\omega)}\right| \leqslant |\eta|^{-L_\delta} \tag{40}$$

in the strip $-1 \leqslant \eta \leqslant 1$, where L_δ is the maximum of the L and L_δ of (ii) and (iii) respectively. The bound (39), in fact, holds throughout the strip. For let us consider the function

$$\frac{(\omega-a)^{L_\delta}\exp(-\omega^2/4\delta)\tilde{F}_-(\omega)}{P_\delta(\omega)}$$

where a is real. This function is regular within and on the boundary defined by the lines ($\eta = \pm 1$; $0 \leqslant \xi/a \leqslant 1$), ($\xi = 0, a$; $-1 \leqslant \eta \leqslant 1$). Its maximum modulus on the boundary is less than or equal to $|\mathrm{i}-a|^{L_\delta}$, and hence

$$\left|\frac{\exp(-\omega^2/4\delta)\tilde{F}_-(\omega)}{P_\delta(\omega)}\right| \leqslant \left|\frac{\mathrm{i}-a}{\omega-a}\right|^{L_\delta}, \qquad 0 \leqslant \xi/a \leqslant 1, \qquad -1 \leqslant \eta \leqslant 1. \tag{41}$$

Thus, letting $|a| \to \infty$, we have (39) for any ω in the strip. Bounding the polynomial by a constant times $(1+|\omega|^{M_\delta})$ and using (39), we see that with C_δ suitably chosen

$$|\tilde{F}_-(\omega)| < C_\delta(1+|\omega|^{M_\delta})\exp\left(\frac{|\omega|^2}{4\delta}\right), \qquad \eta \geqslant -1. \tag{42}$$

From (37) and (iv) we see that a polynomial P independent of δ, whose degree is the maximum of M and N, with no zeros in $\eta \geqslant -1$ can be found, so that $|\tilde{F}_-(\omega)/P(\omega)|$ is less than 1 on the lines $\arg(\omega+\mathrm{i}) = 0, \frac{1}{2}\pi, \pi$. From (42) we have that $\tilde{F}_-(\omega)/P(\omega)$ is certainly of the order $\exp(|\omega+\mathrm{i}|^2/2\delta)$ throughout the region $\eta \geqslant -1$, i.e.

$$\left|\frac{\tilde{F}_-(\omega)}{P(\omega)}\right| < 1, \qquad \arg(\omega+\mathrm{i}) = 0, \tfrac{1}{2}\pi, \pi \tag{43}$$

and

$$\left|\frac{\tilde{F}_-(\omega)}{P(\omega)}\right| < C_\delta \exp\left(\frac{|\omega+\mathrm{i}|^2}{2\delta}\right), \qquad 0 \leqslant \arg(\omega+\mathrm{i}) \leqslant \pi. \tag{44}$$

As (44) holds for all δ, we can apply the Phragmen–Lindelof theorem in the form given by Titchmarsh (1939, § 5.62) to the regions $0 \leqslant \arg(\omega+\mathrm{i}) \leqslant \frac{1}{2}\pi$, $\frac{1}{2}\pi \leqslant \arg(\omega+\mathrm{i}) \leqslant \pi$ to deduce that the bound (43) holds throughout the regions

$$\left|\frac{\tilde{F}_-(\omega)}{P(\omega)}\right| < 1, \qquad \eta \geqslant -1. \tag{45}$$

From (37) and (45) we see that the entire function $\tilde{F}_-(\omega)$ is polynomially bounded and must therefore be a polynomial. The inverse Fourier transform of a polynomial is the sum of the Dirac delta function and its derivatives (this is just the ambiguity in the definition of $F_\pm$). It follows that F has support $[0,\infty)$, being the sum of the distribution F_+ with support $[0,\infty)$ and F_- a distribution with point support at the origin. Thus we have proved that, when the input and output of a system are related as in (1) and satisfy the asymptotic causality condition (8) or (8′), then the system is in fact causal.

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